JOURNAL OF AIRCRAFT
Vol. 46, No. 6, November–December 2009

Engineering Notes

New Improved *g* Method for Flutter Solution

Qiu Ju* and Sun Qin*
Northwestern Polytechnical University, 710072 Xi'an, People's Republic of China

DOI: 10.2514/1.46328

Introduction

THE field of aircraft engineering is entering an era of high technology. Over the past decade there has been a great deal of progress in almost all the subfields in aeronautics. However, in the aeroelastic field, despite the complexity of coupling three distinct engineering disciplines, aerodynamics, structures, and dynamics into a unified aeroelastic analysis capability, computational aeroelasticity has enjoyed a significant number of successes over its course of development. Today, every manned vehicle that flies through our atmosphere undergoes some level of aeroelastic analysis before flight. Also every major unmanned flight vehicle is similarly analyzed.

Furthermore, flutter is a catastrophic aeroelastic phenomenon that must be avoided at all costs, and all flight vehicles must be clear of flutter and many other aeroelastic phenomena in their flight envelope. Flight and wind tunnel testing are two ways to clear a vehicle for flutter, but both are expensive and occur late in the design process. Therefore, engineers rely heavily on computational methods to assess the aeroelastic characteristics of flight vehicles. The successes of computational aeroelasticity are rooted in this aeroelastic characterization process.

Traditionally flutter calculations in frequency domain are performed using either the *K* method or the *P-K* method. The *K* method is generally very fast and quite simple, but it has a downfall in that sometimes the frequency and damping values "loop" around themselves and yield multivalue frequency and damping as a function of velocity. The *K* method solution is only valid when $g = 0$ (g refers to damping) and the structural motion is neutrally stable and matches the aerodynamic motion which is also neutrally stable. The *P-K* method is acknowledged to provide more accurate modal damping values than the *K* method. Gradually, the *P-K* method has become the most widely used method in aeroelastic engineering. In recent years, a *g* method was proposed by P. C. Chen, who uses the analytic property of unsteady aerodynamics and a damping perturbation approach. Although these two methods of *P-K* method and *g* method are different in the equation form, they share the same stability criterion, i.e., eigenroot of aeroelastic equation is solved and a root with positive real part indicates flutter.

In addition, the *g* method uses a reduced-frequency sweep technique to search for the roots of the flutter solution and a predictor-corrector scheme to ensure the robustness of the sweep technique. This *g* method includes a first-order damping term in the flutter equation that is rigorously derived from the Laplace-domain aerodynamics. In the paper, however, the improved *g* method increases a second-order damping term in the flutter equation. It is also valid in the entire reduced frequency domain and up to the second order of damping.

Received 14 July 2009; accepted for publication 13 Aug. 2009.

*P. O. Box 310, School of Aeronautics.

g Method [1]

In Ref. [1], the generalized aeroelastic equations of motion are given by,

$$\left[(V^2/L^2)Mp^2 + K - \frac{1}{2}\rho V^2 Q(ik)\right]\{q\} = 0 \tag{1}$$

where V is the true speed; L is the reference length; and, usually, $L = c/2$; where c is the reference chord; ρ is the air density; q is the generalized coordinates; and M, K, and $Q(ik)$ are the generalized mass, stiffness, and aerodynamic forces matrices, respectively.

Rodden modified Hassig's *P-K* method equation by adding an aerodynamic damping matrix into Eq. (1). The modified *P-K* method equation reads

$$\left[(V^2/L^2)Mp^2 + K - \frac{1}{2}\rho V^2 (Q_I/k)p - \frac{1}{2}\rho V^2 Q_R\right]\{q\} = 0 \tag{2}$$

where Q_R and Q_I are the real part and imaginary part of $Q(ik)$, i.e.,

$$Q(ik) = Q_R + iQ_I \tag{3}$$

By substituting $p = g + ik$ into the third term of Eq. (2), this equation can be rewritten as

$$\left[(V^2/L^2)Mp^2 + K - \frac{1}{2}\rho V^2 (Q_I/k)g - \frac{1}{2}\rho V^2 Q(ik)\right]\{q\} = 0 \tag{4}$$

By comparing Eq. (4) to Eq. (1) it is clearly seen that the extra term $-\frac{1}{2}\rho V^2(Q_I/k)g$ is the added aerodynamic damping matrix.

The first-order Taylor series of $Q(p)$ in terms of k and for small g yields

$$Q(p) \approx Q(ik) + gQ'(ik) \tag{5}$$

where $Q(p)$ is the generalized aerodynamic force. The nondimensional Laplace parameter is p and can be expressed as $p = g + ik$, in which k is the reduce frequency and $g = \gamma k$, in which γ is the transient decay rate coefficient.

Replacing $Q(ik)$ in Eq. (1) by $Q(p)$ of Eq. (5) yields the g method solution:

$$\left[(V^2/L^2)Mp^2 + K - \frac{1}{2}\rho V^2 Q'(ik)g - \frac{1}{2}\rho V^2 Q(ik)\right]\{q\} = 0 \tag{6}$$

Expanding $Q(ik)$ about $ik = 0$ by Taylor's expansion gives

$$Q(ik) = Q(0) + ikQ'(0) + \frac{1}{2}(ik)^2 Q''(0) + \cdots \tag{7}$$

Because all $Q^n(0)$ are real, $Q(ik)$ can be spit into the real and imaginary parts too. It reads

$$Q(ik) = Q_R + iQ_I$$

where

$$Q_R = Q(0) - \frac{1}{2}k^2 Q''(0) + \cdots \tag{8}$$

$$Q_I = kQ'(0) - \frac{1}{6}k^3 Q'''(0) + \cdots \tag{9}$$

Dividing Eq. (9) by k gives the term Q_I/k as

$$Q_I/k = Q'(0) - \frac{1}{6}k^2 Q'''(0) + \cdots \tag{10}$$

Differentiating Eq. (7) with respect to ik gives the term $Q'(ik)$ in Eq. (6) as

$$Q'(ik) = Q'(0) + ikQ''(0) + \cdots \tag{11}$$

Comparison of Eq. (10) with Eq. (11) shows that the equality of Q^I/k and $Q'(ik)$ exists only if $Q(ik)$ is a linear function of k or at $k = 0$. This proves that the added aerodynamic damping matrix in Eq. (4) is valid only if one of these conditions is satisfied. In fact, if $Q(ik)$ is highly nonlinear, the P-K method may produce unrealistic roots because of the error from the differences between Eqs. (10) and (11).

Substituting $p = g + ik$ into Eq. (6) yields a second-order linear system in terms of g:

$$[g^2 A + gB + C]\{q\} = 0 \tag{12}$$

where $A = (V/L)^2 M$, $B = 2ik(V/L)^2 M - \frac{1}{2}\rho V^2 Q'(ik) + (V/L)Z$, and $C = -k^2(V/L)^2 M + k - \frac{1}{2}\rho V^2 Q(ik) + ik(V/L)Z$. Generally, suppose that a modal structural dampling matrix Z is equal to zero. Equation (12) is rewritten in a state-space form:

$$[D - gI]\{X\} = 0 \tag{13}$$

where

$$D = \begin{bmatrix} 0 & I \\ -A^{-1}C & -A^{-1}B \end{bmatrix}$$

and $\{X\}$ is the right eigenvector of the state-space equation.

Then, a reduced-frequency-sweep technique is used, which searches for the condition $\mathrm{Im}(g) = 0$ and solves for the eigenvalues of D in terms of g. And the flutter frequency ω_f and damping 2γ are computed by

$$\omega_f = k(V/L) \tag{14}$$

$$2\gamma = 2[Re(g)/k] \tag{15}$$

Especially for $k = 0$ an alternative form of Eq. (15) is used [2]:

$$2\gamma = \frac{Re(g)(L/V)}{\ln(2)} \tag{16}$$

Euler Predictor-Corrector Method

The idea behind the predictor-corrector methods is to use a suitable combination of an explicit and an implicit technique to obtain a method with better convergence characteristics. The method, referred to as the Euler method is given below.

$$y_{n+1} = y_n + hf(x_n, y_n) \tag{17}$$

$$y_0 = y(x_0), \qquad n = 1, 2, \cdots, n \tag{18}$$

In the present study the predictor-corrector method described earlier is an explicit method.

Iterative Procedure

The predictor predicts the eigenvalues at $|k + \Delta k|$ by a linear extrapolation from the eigenvalues and their derivatives at k:

$$g_p(k + \Delta k) = g(k) + \Delta k \frac{\mathrm{d}g}{\mathrm{d}k} \tag{19}$$

where g_p is defined as the predicted eigenvalue, $\mathrm{d}g/\mathrm{d}k$ can be obtained by using the orthogonality property of the left and right eigenvector of Eq. (13). This leads to

$$\frac{\mathrm{d}g}{\mathrm{d}k} = \left(Y^T \frac{\mathrm{dD}}{\mathrm{d}k} X\right) \Big/ Y^T X \tag{20}$$

where Y and X are the left and right eigenvectors of Eq. (13), respectively, and

$$\frac{\mathrm{dD}}{\mathrm{d}k} = \begin{bmatrix} 0 & 0 \\ -A^{-1}\frac{\mathrm{d}C}{\mathrm{d}k} & -A^{-1}\frac{\mathrm{d}B}{\mathrm{d}k} \end{bmatrix} \tag{21}$$

Once g_p is given by the predictor, g_p is used as the baseline eigenvalues for sorting the computed eigenvalues at $|k + \Delta k|$ defined as g_c. The maximum norm of the error between g_p and g_c for all eigenvalues is also computed. If it exceeds a certain level, the predictor could potentially introduce incorrect eigenvalue tracking results due to rapid changes of the eigenvalues with respect to k in the reduced-frequency-sweep process. In this case, the corrector is activated.

The corrector reduces the size of Δk by a factor, for instance, and recomputes g_p and g_c at $(k + \Delta k/100)$ as well as the maximum norm of the error. This process repeats until the maximum norm of the error is below a certain level. However, numerical experience shows that when the corrector is activated, this condition can be satisfied by reducing the size of Δk only once. Therefore, the corrector normally would not increase the computational time significantly. It serves only as a fail-safe scheme.

Novel *g* Method

To enhance the solution accuracy, now the highly nonlinear term of $Q(ik)$ is considered. For Eq. (5), expanding $Q(p)$ by Taylor's second-order expansion gives

$$Q(p) \approx Q(ik) + gQ'(ik) + \frac{1}{2}g^2 Q''(ik) \tag{22}$$

Likewise, substituting $p = g + ik$ and Eq. (22) into Eq. (1) yields

$$\left\{\left[\frac{V^2}{L^2}M - \frac{1}{4}\rho V^2 Q''(ik)\right]g^2 + \left[2ik\frac{V^2}{L^2}M - \frac{1}{2}\rho V^2 Q'(ik)\right]g - \frac{V^2}{L^2}Mk^2 + K - \frac{1}{2}\rho V^2 Q(ik)\right\}\{q\} = 0 \tag{23}$$

The preceding equation is written as

$$[g^2 A' + gB + C]\{q\} = 0 \tag{24}$$

where $A' = (V/L)^2 M - \frac{1}{4}\rho V^2 Q''(ik)$, $B = 2ik(V/L)^2 M - \frac{1}{2}\rho V^2 Q'(ik) + (V/L)Z$, and $C = -k^2(V/L)^2 M + k - \frac{1}{2}\rho V^2 Q(ik) + ik(V/L)Z$. Especially, by comparing A' to A, it is clearly shown that the extra term $-\frac{1}{4}\rho V^2 Q''(ik)$ is the added second-order aerodynamic damping matrix.

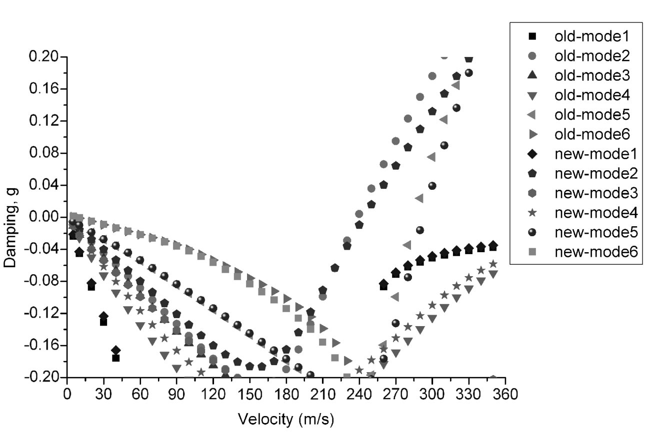

Fig. 1 *V-g* curve.

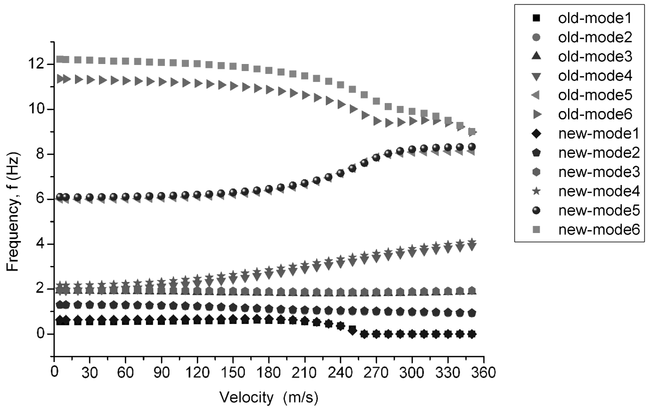

Fig. 2 *V-f* curve.

Furthermore, Eq. (13) is expressed as

$$[\mathrm{D}' - gI]\{X\} = 0 \tag{25}$$

where

$$\mathrm{D}' = \begin{bmatrix} 0 & I \\ -A'^{-1}C & -A'^{-1}B \end{bmatrix} \tag{26}$$

and Eq. (21) is changed to

$$\frac{\mathrm{d}\mathrm{D}'}{\mathrm{d}k} = \begin{bmatrix} 0 & 0 \\ -A^{-1}\frac{\mathrm{d}C}{\mathrm{d}k} - \frac{\mathrm{d}A^{-1}}{\mathrm{d}k}C & -A^{-1}\frac{\mathrm{d}B}{\mathrm{d}k} - \frac{\mathrm{d}A^{-1}}{\mathrm{d}k}B \end{bmatrix} \tag{27}$$

where

$$\frac{\mathrm{d}A^{-1}}{\mathrm{d}k} = -A^{-1}\frac{\mathrm{d}A}{\mathrm{d}k}A^{-1} \tag{28}$$

Then, use the same method to solve for the eigenvalues of D in terms of g.

Example Application

According to some T-tail structure, DMAP language is used to extract the generalized mass, stiffness, and aerodynamic matrix from the standard flutter solution sequence of NASTRAN. This flutter problem is solved by use of the g method and the improved g method. The results are demonstrated in the $V - g$ and $V - f$ diagrams (Figs. 1 and 2).

As can be seen from the preceding figures, two methods present the same bending-torsion coupling modes that lead to flutter mechanisms. However, aeroelastic flutter speeds and frequencies have some differences. The flutter speeds are 235 m/s by the new method and 245 m/s by the old method. The corresponding flutter frequencies are 1.012 and 1.044 Hz.

Conclusions

1) The standard g method generalizes the conventional k and p-k methods and provides more reliable damping for flutter predictions. This method yields an exact flutter equation solution accurate up to the first order of damping. This solution can be derived from the Laplace-domain aerodynamics and is valid throughout the entire reduced-frequency domain. Although recent developments in the field of g method analysis hold promise for a more robust and efficient method for the solution of flutter problems, the present study shows the improved g method, including a second-order damping term in the flutter equation, and that the great progress which has been made on these methods can be applied to flutter problems. Again, the computational accuracy is enhanced in theory.

2) The old and new g methods use a reduced-frequency sweep technique to search for the roots of the flutter solution and a predictor-corrector scheme to ensure the robustness of the sweep technique. The solution algorithm of the g method is proven to be efficient and robust, as indicated by the results of the selected test case.

3) Computational aeroelasticity continues to play a critical role in the development of modern air vehicles. The current suite of linear aeroelastic serves as the mainstay for aeroelstic analysis and they form a solid base on which to build future developments. Unfortunately, modern aircraft systems continually uncover aeroelastic issues that cannot be effectively predicted by these methods, based on linear aerodynamic forces and small aeroelastic perturbation. Therefore, it is imperative that higher-order aeroelastic methods continue to be developed and refined. In the improved g method, when the higher-order aerodynamic damping is added to the flutter solution, it is fully suitable for the future development trend.

4) Finally, the application example demonstrates the new g method is really a computational aeroelasticity technology in innovative or novel ways. This approach to the analysis provides the designers with an improved confidence that the design will be free from aeroelastic anomalies throughout the flight envelope.

References

[1] Chen, P. C., "Damping Perturbation Method for Flutter Solution: The g-Method," *AIAA Journal*, Vol. 38, No. 9, Sept. 2000, pp. 1519–1521. doi:10.2514/2.1171

[2] Patil, M. J., "Nonlinear Aeroelastic Analysis, Flight Dynamics, and Control of a Complete Aircraft", Ph.D. Dissertation, School of Aerospace Engineering, Georgia Institute of Technology, Atlanta, GA, 1999.